INTEGRATION

Integration can be developed into two ways which can be shown to be fundamentally equivalent:

(i) ***as the reverse of differentiation ,***
(ii) ***as a limit of a sum .***

Integration is a process of finding a function from its derived function and it reverses the operation of differentiation.

We have:

(a) Indefinite integrals

These integrals arise basically from the approach (i) above .

For example, we know that $\frac{d}{dx}\left(x^4\right) = 4x^3$, and that we can write this information in integral form as

$\int 4x^3 dx = x^4 + C$, where C is arbitrary constant called the ***constant of integration.***

Generally:

If $\frac{d}{dx}(g(x)) = f(x)$, then $\int f(x) dx = g(x) + C$.

$f(x)$ is called the ***integrand.***

(b) Definite integrals

These integrals involve limits:

$$\int_2^3 4x^3 dx = \left[x^4\right]_2^3 = 3^4 - 2^4 = 72.$$

We obtain a definite answer – there is ***no arbitrary constant!***

Generally:

If $\frac{d}{dx}(g(x)) = f(x)$, then $\int_a^b f(x) dx = [g(x)]_a^b = g(b) - g(a)$.

The last result is called **Fundamental Theorem of Calculus** (Theorem of Leibniz-Newton).

Some basic rules:

$$\int \{f(x) + g(x)\}dx = \int f(x)dx + \int g(x)dx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int k dx = kx + C$$

$$\int \frac{1}{x} dx = \ln x + C$$

$$\int e^x dx = e^x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \frac{1}{\cos^2 x} dx = \tan x + C$$

$$\int \frac{1}{\sin^2 x} dx = -\cot x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Worked examples

1) If $\dfrac{dy}{dx} = x^4 + \dfrac{1}{x3}$, find y.

Solution:

$$y = \int \left(x^4 + \frac{1}{x^3} \right) dx = \frac{x^{4+1}}{4+1} + \frac{x^{-3+1}}{-3+1}.$$

$$= \frac{x^5}{5} + \frac{x^{-2}}{-2} = \frac{x^5}{5} - \frac{1}{2x^2} + C.$$

2) $\int (3x-2)\left(4+x^2\right) dx = \int \left(12x + 3x^3 - 8 - 2x^2\right) dx$

$$= \int 3x^3 dx - \int 2x^2 dx + \int 12x dx - \int 8 dx$$

$$= 3\frac{x^4}{4} - 2\frac{x^3}{3} + 12\frac{x^2}{2} - 8x + C.$$

3) $\int \frac{x^3+2}{\sqrt{x}} dx = \int \left(\frac{x^3}{\sqrt{x}} + \frac{2}{\sqrt{x}} \right) dx = \int \left(x^{\frac{5}{2}} + 2x^{\frac{-1}{2}} \right) dx$

$$= \int x^{\frac{5}{2}} dx + 2\int x^{\frac{-1}{2}} dx = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= \frac{2}{7} x^{\frac{7}{2}} + 4x^{\frac{1}{2}} + C.$$

4) $\int \left(3x^2 + \sqrt[3]{x} - \frac{5}{x^2} - \frac{4}{x} \right) dx$

$$= \int 3x^2 dx + \int x^{\frac{1}{3}} dx - \int 5x^{-2} dx - \int \frac{4}{x} dx$$

$$= 3\frac{x^3}{3} + \frac{3}{4} x^{\frac{4}{3}} + 5x^{-1} - 4\ln x$$

$$= x^3 + \frac{3}{4} x^{\frac{4}{3}} + \frac{5}{x} - 4\ln x + C.$$

5) Find the equation of the curve with gradient $4 - 3x^2$ which passes through the point $(4,1)$.

Solution:

The gradient of the curve is $\dfrac{dy}{dx}$, and so $\dfrac{dy}{dx} = 4 - 3x^2$.

Since $y = \int \left(4 - 3x^2\right)dx = 4x - 3\dfrac{x^3}{3} + C = 4x - x^3 + C$.

But the curve passes through the point $(4,1)$, and so $y = 1$ when $x = 4$.

Substituting these values into our expression for y gives:

$1 = 4.4 - 4^3 + C \Rightarrow 1 = 16 - 64 + C \Rightarrow C = 48$.

Thus the equation of the curve is $y = 4x - x^3 + 48$.

6) The speed v of a body at time t is given by $v = 2t - \dfrac{5}{t^2}$.

Let s is a displacement of the body at time t and it is known that $s = \dfrac{3}{2}$ when $t = 2$. Find s in term of t.

Solution:

We know that $v = \dfrac{ds}{dt}$, and so we are given that $\dfrac{ds}{dt} = 2t - \dfrac{5}{t^2}$.

But $s = \dfrac{3}{2}$ when $t = 2$, and we obtain

$\dfrac{3}{2} = 4 + \dfrac{5}{2} + C \Rightarrow C = -5$.

7) Evaluate the definite integral $\int_0^1 x(\sqrt{x} + 3)dx$.

Solution:

$$x\left(\sqrt{x} + 3\right) = x\left(x^{\frac{1}{2}} + 3\right) = x^{\frac{3}{2}} + 3x,$$

$$\int_0^1 \left(x^{\frac{3}{2}} + 3x \right) dx = \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 3\,\frac{x^2}{2} \right]_0^1 = \left[\frac{2}{5}\, x^{\frac{5}{2}} + \frac{3}{2}\, x^2 \right]_0^1$$

$$= \left(\frac{2}{5} + \frac{3}{2} \right) - 0 = \frac{19}{10}$$

Exercises

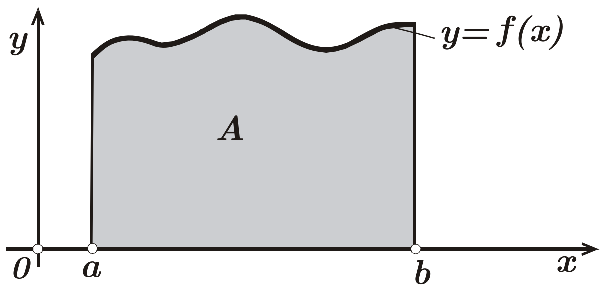
1) Integrate the following functions with respect to x:

a) $6x^2$; $\quad x^{\frac{2}{3}}$; $\quad \frac{5}{x^3}$; $\quad \sqrt[4]{x}$; $\quad \frac{1}{x^7}$;

b) $4x^2 - \sqrt{x} + \frac{6}{x^{-3}}$; $\quad \frac{(\sqrt{x} - 2)}{x^3}$

c) $\frac{1}{3}x + \sqrt[3]{x} - \frac{2}{\sqrt{x}} - 5$;

• Areas under curves



Suppose that we wish to find the area bounded by the curve $y = f(x)$, the x axis and the lines $x = a$; $x = b$.

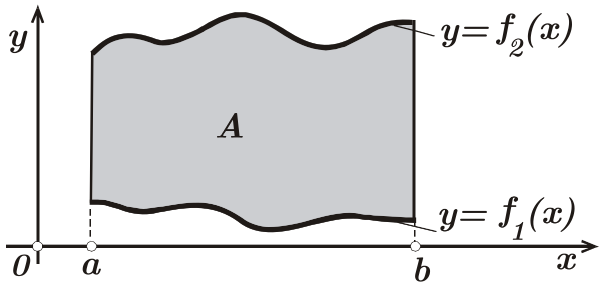
Using the method to split the area into thin vertical strips, the sum of the areas of these rectangular strips then gives an approximate value for the required area.

This result is given by the definite integral

$A = \int_a^b y dx$, where $x = a$ is lower limit, $x = b$ is bound (upper) limit.

The area bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ can be calculated by the next definite integral

$$A = \int_a^b \left(f_2 - f_1 \right) dx,$$ where the sketch is



Note

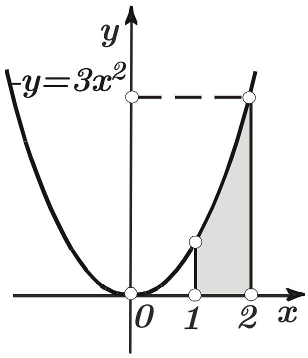
We can apply this formula only if $f(x)$,being integrated, is definite for $\forall\, x \in [a, b]$.

Example1.

Find the area of the region bounded by the curve $y = 3x^2$, the x-axis and the lines $x = 1$, $x = 2$.

Solution:

The sketch is shown – it is essential !



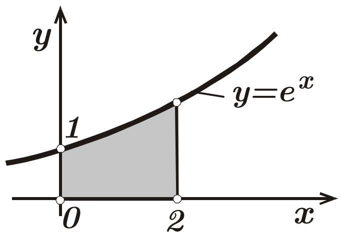
We find that

$$A = \int_1^2 3x^2 dx = \left[x^3 \right]_1^2 = 8 - 1 = 7\,.$$

Example2.

Find the area bounded by the curve $y = e^x$, x - axis, y - axis and the line $x = 2$.

Solution:



$$A = \int_0^2 e^x dx = \left[e^x\right]_0^2 = e^2 - e^0 = e^2 - 1.$$

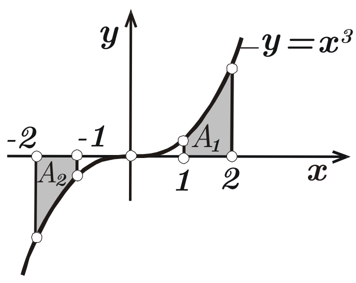
The meaning of a negative result

Consider the area bounded by $y = x^3$, x - axis, y - axis and the lines:

(i) $x = 2, \quad x = 1;$

(ii) $x = -2, \quad x = -1.$

Solution:



(i) Required area is A_1 which is

$$A_1 = \int_1^2 x^3 dx = \left[x^4\right]_1^2 = 2^4 - 1^4 = 15.$$

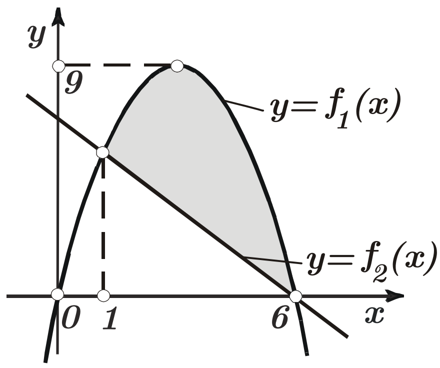
(ii) Required area is A_2. By symmetry $A_2 = 15$ (the curve is symmetric about the origin, so the two areas are equal).

Example3.

Find the area of the region enclosed by the curve $y = 6x - x^2$ and the line $y = 6 - x$.

Solution:

Solve the equations simultaneously to find the point of intersection:



$$\begin{cases} y = 6x - x^2 \\ y = 6 - x \end{cases} \Rightarrow$$

$$x^2 - 7x + 6 = 0,$$

$$x_2 = 1 \qquad y_2 = 5,$$

$$x_2 = 6 \qquad y_2 = 0,$$

$$A = \int_1^6 (y_1 - y_2) = \int_1^6 \left[\left(6x - x^2\right) - (6 - x)\right] dx$$

$$= \int_1^6 \left(-x^2 + 7x - 6\right) dx = \left[\frac{-x^3}{3} + \frac{7x^2}{2} - 6x\right]_1^6$$

$$= \left[\frac{-6^3}{3} + \frac{7.6^2}{2} - 6.6\right] - \left[\frac{-1^3}{3} + \frac{7^2}{2} - 6\right] = \frac{125}{6} .$$

• Integration by substitution

Suppose, we have to find $\int \cos 4x \, dx$.

We could proceed as follows :

Let $F(x) = \int \cos(4x) \, dx$, then $\dfrac{dF}{dx} = \cos(4x)$.

Let $4x = u$, then $\dfrac{du}{dx} = 4$ and so

$\dfrac{dx}{du} = \dfrac{1}{4} \Rightarrow \cos(4x) = \cos u.$

Use the rule " function of a function" (chain rule) :

$$\frac{dF}{du} = \frac{dF}{dx}\cdot\frac{dx}{du} = (\cos 4x)\frac{1}{4} = \cos u.\frac{1}{4},$$

$$F = \int \frac{1}{4}\cos u du = \frac{1}{4}\sin u + C,$$

$F(x) = \frac{1}{4}\sin 4x + C$ (finally going back to original variable x).

Thus $\int \cos 4x dx = \frac{1}{4}\sin 4x dx + C.$

For the above integral the following layout and technique is an equivalent procedure and considerably more convenient.

Find $\int \cos 4x dx.$

Let $u = 4x \Rightarrow \frac{du}{dx} = 4 \Rightarrow dx = \frac{du}{4}.$

So $\int \cos 4x dx$

$$= \int \cos u d\frac{u}{4}$$

$$= \frac{1}{4}\int \cos u du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin 4x + C.$$

Don't forget to convert back to the original variable!

So

$$\int f[g(x)]g'(x)dx = \int f[g(x)]dg(x) = \int f(u)du$$
$$= F(u) + C = F[g(x)] + C.$$

In particular

$$\int f(x+a)dx = \int f(x+a)d(x+a) = \int f(u)du$$
$$= F(u) + C = F(x+a) + C.$$

Example1.

$$I = \int x^2 \sqrt{x^3 + 2}\, dx\,.$$

Since $\frac{d}{dx}\left(x^3 + 2\right) = 3x^2$, let $u = \left(x^3 + 2\right)$,

$\frac{du}{dx} = 3x^2$, $\frac{du}{dx} = 3x^2$, $x^2 dx = \frac{du}{3}$.

So

$$I = \int \sqrt{x^3 + 2}\left(x^2 dx\right)$$

$$= \int \sqrt{u}\left(\frac{du}{3}\right) \quad = \int u^{\frac{1}{3}} du = \frac{1}{3}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{9} u^{\frac{3}{2}} + C = \frac{2}{9}\left(x^3 + 2\right)^{\frac{3}{2}} + C\,.$$

Example2.

$$J = \int 4xe^{x^2}\, dx\,.$$

Since $\frac{d}{dx}\left(x^2\right) = 2x$, let $u = x^2$, $\frac{du}{dx} = 2x$, $xdx = \frac{du}{2}$.

So

$$J = \int 4e^{x^2}\ (xdx)$$

$$= \int 4e^u \frac{du}{2}$$

$$= 2e^u + C = 2e^{x^2} + C\,.$$

Example3.

$$K = \int \frac{1}{\sqrt{3x - 1}}\, dx\,.$$

Let $u = 3x - 1$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$.

So

$$K = \int \frac{1}{\sqrt{u}} \frac{du}{3} = \frac{1}{3}\int u^{\frac{-1}{2}} du$$

$$= \frac{1}{3} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}\sqrt{u} + C = \frac{2}{3}\sqrt{3x-1} + C\,.$$

Example4.

$$L = \int \frac{x}{4x^2-1}\, dx\ .$$

Let $u = 4x^2 - 1,\ \frac{du}{dx} = 8x\ ,\quad xdx = \frac{du}{8}\ .$

So

$$L = \int \frac{1}{u}\left(\frac{du}{8}\right) = \frac{1}{8}\int \frac{1}{u} du = \frac{1}{8}\ln u + C$$

$$= \frac{1}{8}\ln\left(4x^2 - 1\right) + C.$$

Exercises

1) $\int \sin(x-3)\, dx$ **2)** $\int \cos(x+2)\, dx$ **3)** $\int (x-4)^6\, dx$

4) $\int \sqrt{x+3} dx$ **5)** $\int \frac{dx}{\sqrt{3x+2}}$ **6)** $\int \frac{dx}{x+4}$

7) $\int e^{2x} dx$ **8)** $\int \frac{dx}{\cos^2 3x}$ **9)** $\int \frac{dx}{\sin^2 \frac{x}{4}}$

10) $\int \frac{dx}{1+9x^2}$ **11)** $\int \frac{x}{\sqrt{2x-1}}\, dx$ **12)** $\int \frac{4x}{2-x^2}\, dx$

Note

If you find difficulty in picking the correct substitution, then you may succeed by the much "rougher" argument:

Put $u =$ the most complicated part of the integral!

Always use a substitution, though if you are in doubt, since the substitution procedure is simple enough in it self!

Do not think that any integral can be done by substitution – this is **not** so!

Specially Important Integral

Remember

$$\frac{d}{dx}\{\ln(f(x))\} = \frac{f'(x)}{f(x)} \Rightarrow \int \frac{f'(x)}{f(x)}dx = \ln x + C \ !$$

We can do $I = \int \frac{x}{4x^2 - 1}dx$ as follows

$$I = \int \frac{x}{4x^2 - 1}dx = \frac{1}{8}\int \frac{8x}{4x^2 - 1}dx = \frac{1}{8}\ln\left(4x^2 - 1\right) + C$$

(multiplying and dividing by the constant 8, the numerator becomes derivative of the denominator).

Further examples

1) $\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\ln(\cos x) + C$

(because the numerator is the derivative of the denominator; or we can substitute $u = \sin x$).

2) $\int \frac{x+1}{x^2 + 2x - 4}dx = \frac{1}{2}\int \frac{2x+2}{x^2 + 2x - 4}dx$

$= \frac{1}{2}\ln\left(x^2 + 2x - 4\right) + C$

(multiplying numerator and denominator by 2, since numerator becomes derivative of denominator ; or use the substitution $u = x^2 + 2x - 4$).

• Definite integral by substitution

We have **always** to change the limits.
It is then not necessary to convert back to the original variable at the end of the integration process.

Examples

1) Find $\displaystyle\int_0^1 x\left(3x^2-2\right)^8 dx$.

Let $u = 3x^2 - 2$, $\dfrac{du}{dx} = 6x$, $xdx = \dfrac{du}{6}$.

Change the limits

$$\begin{cases} x = 0 \rightarrow u = -2 \\ x = 1 \rightarrow u = 1 \end{cases}$$

Then we obtain

$$\int_0^1 x\left(3x^2-2\right)^8 dx = \int_0^1 \left(3x^2-2\right)^8 xdx$$

$$= \int_{-2}^1 u^8 \frac{du}{6} = \frac{1}{6}\left[\frac{u^9}{9}\right]_{-2}^1 = \frac{1}{54}\left[u^9\right]_{-2}^1$$

$$= \frac{1}{54}\left(1^9 - (-2)^9\right) = \frac{1}{54}(513) = \frac{513}{54}.$$

2) Find $\int_{0}^{\frac{\pi}{6}} \sin^4 x \cos x dx$.

Let $u = \sin x$, $\frac{du}{dx} = \cos x$, $\cos x dx = du$.

Now change the limits $\begin{cases} x = 0 \to u = 0 \\ x = \frac{\pi}{6} \to u = \frac{1}{2} \end{cases}$

So

$$\int_{0}^{\frac{\pi}{6}} \sin^4 x \,(\cos x dx)$$

$$= \int_{0}^{\frac{1}{2}} u^4 du = \left[\frac{u^5}{5}\right]_0^{\frac{1}{2}} = \frac{1}{5}\left[\left(\frac{1}{2}\right)^5 - 0^5\right] = \frac{1}{160}.$$

3) Find the following definite integrals

a) $\int_{0}^{4} \sqrt{2x+5}dx$ **b)** $\int_{0}^{\frac{\pi}{4}} \frac{\cos\theta}{\sin^3\theta}d\theta$ **c)** $\int_{0}^{\frac{\pi}{4}} \sec^4 x \tan x dx$

d) $\int_{1}^{2} t^2 e^{-t^3} dt$ **e)** $\int_{1}^{2} \frac{t^2}{1+t^3} dt$ **f)** $\int_{0}^{3} t\sqrt{4-t^2}dt$

k) $\int_{0}^{\frac{\pi}{3}} \sin t \sqrt{\cos t}dt$ **m)** $\int_{1}^{2} \frac{x}{\sqrt{3+x^2}}dx$ **n)** $\int_{2}^{3} x\left(1-x^2\right)dx$

4) Solve

a) $\int \sin 2x dx$ **b)** $\int \cos 3x dx$ **c)** $\int e^{4x} dx$

d) $\int \frac{1}{1+2x} dx$ **e)** $\int (4x-1)^7 \, dx$ **f)** $\int \frac{dx}{(1-x)^2}$

k) $\int \frac{dx}{3x-4}$ **m)** $\int \frac{dx}{\sqrt{5x+3}}$ **n)** $\int \sqrt{5-x} dx$

• Further substitutions. Definite integral by substitution

Unfortunately, it isn't possible to give a general rule which will always work when choosing a substitution.

As usual in mathematics, it is practice and experience which are necessary.

However, often if you substitute $u =$ "the function involved in the most complicated or difficult part of the integral" , the substitution will work.

Examples

1) Find $I = \int \frac{x}{\sqrt{3x+1}} dx$.

Let $u = 3x + 1,\quad \frac{du}{dx} = 3,\quad dx = \frac{du}{3}.$

Also

$3x = u - 1$ and $x = \frac{1}{3}(u-1).$

This gives

$$I = \int \frac{\frac{1}{3}(u-1)}{\sqrt{u}} \frac{du}{3} .$$

Everything is converted to u ***!***

$$I = \frac{1}{9}\int \frac{u-1}{\sqrt{u}}du = \frac{1}{9}\int\left(\frac{u}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}}\right)du$$

$$= \frac{1}{9}\int\left(u^{\frac{1}{2}} - u^{\frac{-1}{2}}\right)du$$

$$= \frac{1}{9}\left[\int u^{\frac{1}{2}}du - \int u^{\frac{-1}{2}}du\right]$$

$$= \frac{1}{9}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$$

$$= \frac{1}{9}\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right] + C$$

$$= \frac{2}{9}u^{\frac{1}{2}}\left[\frac{u}{3} - 1\right] + C = \frac{2\sqrt{u}}{9}\left(\frac{u-3}{3}\right) + C$$

$$= \frac{2}{27}\sqrt{3x+1}\Big(\big(3x+1\big)-3\Big) + C$$

$$= \frac{2}{27}\sqrt{3x+1}\big(3x-2\big) + C\,\textbf{.}$$

Always try to express the answer in its simplest form!

3) Find $J = \int_0^{\frac{\pi}{3}} \cos^4 x \sin x dx\,.$

Let $u = \cos x\,,\quad \frac{du}{dx} = -\sin x\,,\quad \sin x dx = -du\,.$

Also change the limits $\begin{cases} x = 0 \to u = 1 \\ x = \dfrac{\pi}{3} \to u = \dfrac{1}{2} \end{cases}$

Then

$$J = \int_0^{\frac{\pi}{3}} \cos^4 x\,(\sin x dx)$$

$$= \int_1^{\frac{1}{2}} u^4\,(-du)$$

$$= -\int_1^{\frac{1}{2}} u^4 du = -\left[\frac{u^5}{5}\right]_1^{\frac{1}{2}} = -\frac{1}{5}\left[\left(\frac{1}{2}\right)^5 - 1^5\right] = \frac{31}{160}.$$

4) Find $K = \int_0^2 x\,(x-2)^5 dx$.

Let $u = x - 2,\quad \dfrac{du}{dx} = 1,\quad dx = du,$

also $x = u + 2$.

Change the limits

$$\begin{cases} x = 0 \to u = -2 \\ x = 2 \to u = 0. \end{cases}$$

Then

$$K = \int_{-2}^{0} (u+2)\,u^5 du$$

$$= \int_{-2}^{0} \left(u^6 + 2u^5\right) du$$

$$= \int_{-2}^{0} u^6 du + 2\int_{-2}^{0} u^5 du$$

$$= \frac{u^7}{7}\bigg|_{-2}^{0} + \cancel{2} \cdot \frac{u^6}{\cancel{6}_3}\bigg|_{-2}^{0}$$

$$= \left(0 - \frac{(-2)^7}{7}\right) + \left(0 - \frac{(-2)^6}{3}\right)$$

$$= \frac{108}{7} + \frac{64}{3} = \frac{772}{21} \,.$$

• Trigonometric and Hyperbolic substitutions

There are many trigonometric identities which can help us to simplify certain expressions.

For example

$\sin^2 x + \cos^2 x = 1$ $\qquad$ $1 + \tan^2 x = \sec^2 x$

$\cosh^2 x - \sinh^2 x = 1$ $\qquad$ $1 - \tanh^2 x = \operatorname{sech}^2 x$

Consider the integral

$$I = \int \frac{1}{\sqrt{a^2 - x^2}}\, dx \,.$$

If we substitute $x = a\sin\theta$, so

$$dx = a\cos\theta d\theta \,, \quad \frac{dx}{d\theta} = a\cos\theta \,.$$

Then $a^2 - x^2 = a^2 - (a\sin\theta)^2$

$$= a^2 - a^2\sin^2\theta$$

$$= a^2\left(1-\sin^2\theta\right)$$

$$= a^2\cos^2 x.$$

Now

$$\sqrt{a^2 - x^2} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

And also $\sin\theta = \frac{x}{a}$ gives $\theta = \arcsin\left(\frac{x}{a}\right)$.

So

$$I = \int \frac{1}{a\cos\theta}(a\cos\theta d\theta) = \int 1 d\theta = \theta + C$$

$$= \arcsin\left(\frac{x}{a}\right) + C.$$

Further example.

Find $K = \int_0^1 \frac{1}{\left(1+x^2\right)^2}dx.$

Let $x = \tan\theta$, $\frac{dx}{d\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$, $dx = \sec^2\theta d\theta.$

Now $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta.$

Also $x = \tan\theta$ gives $\theta = \tan^{-1} x$,

and
$$\begin{cases} x = 0 \to \theta = 0 \\ x = 1 \to \theta = \frac{\pi}{4}. \end{cases}$$

So we have

$$K = \int_0^{\frac{\pi}{4}} \frac{1}{\sec^4\theta}\sec^2\theta d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 \theta d\theta \quad = \int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 2\theta)d\theta \quad = \frac{1}{2}\left\{\int_0^{\frac{\pi}{4}} 1 d\theta + \int_0^{\frac{\pi}{4}} \cos 2\theta d\theta\right\}$$

$$= \frac{1}{2}\left\{[\theta]_0^{\frac{\pi}{4}} + \left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi}{4} - 0\right) + \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right)\right\}$$

$$= \frac{1}{8}(\pi + 2).$$

Note

The obvious substitution $u = 1 + x^2$ doesn't work (you can try).
However, because $1 + \tan^2\theta = \sec^2\theta$, we can simplify the $\left(1 + x^2\right)$ by substituting $x = \tan\theta$.

• Use of Standard Integrals (S.I.)

When using the S.I. sheet , your integral must be in exactly the same form as that given on the sheet. It may be necessary to do some algebraic manipulation , before the standard form can be used.

Example1.

$$I = \int \frac{1}{\sqrt{9 - 16x^2}} dx \ .$$

We have $\sqrt{9 - 16x^2} = \sqrt{16\left(\frac{9}{16} - x^2\right)} = 4\sqrt{\left(\frac{3}{4}\right)^2 - x^2}$.

Then

$$I = \int \frac{1}{4\sqrt{\left(\frac{3}{4}\right)^2 - x^2}} = \frac{1}{4}\int \frac{1}{\sqrt{\left(\frac{3}{4}\right)^2 - x^2}}$$

$$= \frac{1}{4}\arcsin\left(\frac{x}{\left(\frac{3}{4}\right)}\right) + C\,.$$

Note

We could do this integral by substituting $x = \frac{3}{4}\sin\theta\,.$

Example2.

$$J = \int_0^{\frac{2}{3}} \frac{3}{4 + 9x^2}dx\,.$$

We have $4 + 9x^2 = 9\left(\frac{4}{9} + x^2\right) = 9\left(\left(\frac{2}{3}\right)^2 + x^2\right),$

so

$$J = \int_0^{\frac{2}{3}} \frac{3}{9\left(\left(\frac{2}{3}\right)^2 + x^2\right)}dx = \frac{1}{3}\int_0^{\frac{2}{3}} \frac{1}{\left(\frac{2}{3}\right)^2 + x^2}dx$$

$$= \frac{1}{3}\left[\frac{1}{\left(\frac{2}{3}\right)}\arctan\left(\frac{x}{\left(\frac{2}{3}\right)}\right)\right]_0^{\frac{2}{3}}$$

$$= \frac{1}{3}.\frac{3}{2}\left[\arctan\left(\frac{3x}{2}\right)\right]_0^{\frac{2}{3}} = \frac{1}{2}\left(\arctan 1 - \arctan 0\right) = \frac{1}{2}.\frac{\pi}{4} = \frac{\pi}{8}.$$

Example3.

$$K = \int \frac{2}{\sqrt{4t^2 + 25}} dt .$$

We have $\sqrt{4t^2 + 25} = \sqrt{4\left(t^2 + \frac{25}{4}\right)} = 2\sqrt{t^2 + \left(\frac{5}{2}\right)^2}$,

so $K = \int \frac{2}{2\sqrt{t^2 + \left(\frac{5}{2}\right)^2}} dt = \int \frac{1}{\sqrt{t^2 + \left(\frac{5}{2}\right)^2}} dt = \arcsin\left(\frac{t}{\left(\frac{5}{2}\right)}\right) + C$

Often it is more convenient to use the logarithmic form $\ln\left(t + \sqrt{t^2 + \left(\frac{5}{2}\right)^2}\right)$.

Exercises

1) Find $\int 3x\,(2x-1)^7 dx$, using the substitution $u = 2x - 1$.

2) Find $\int \frac{1}{\left(1 + x^2\right)^{\frac{3}{2}}} dx$, using the substitution $x = \tan\theta$.

3) Find $\int_0^{\sqrt{2}} \frac{1}{\left(4 - x^2\right)^{\frac{5}{2}}} dx$, using the substitution $x = 2\sin\theta$.

4) Find $\int \sqrt{\frac{x}{1-x}}\, dx$, using the substitution $x = \sin^2\theta$.

5) Find $J = \int_{\frac{1}{3}}^{2} x\sqrt{3x+4}dx$; $J = \int_{0}^{\frac{\pi}{3}} \frac{\tan x}{\sqrt{\sec x}} dx$.

6) Use Standard Integrals to find:

a) $\int \frac{1}{\sqrt{9-4x^2}}dx$ **b)** $\int \frac{3}{1+9x^2}dx$ **c)** $\int \frac{5}{25+16x^2}dx$

d) $\int \frac{2}{\sqrt{4x^2+7}}dx$ **f)** $\int \frac{1}{\sqrt{4-(x+2)^2}}dx$ (put $t = x+2$)

• Integration of Rational Function

A Rational Function is an algebraic fraction with polynomials in the numerator and the denominator,

e.g. $$\frac{ax^2+bx+c}{px^3+qx+rx+m}.$$

For integrating a Rational Function we have to express it in partial fractions. If the degree of the numerator is less than the degree of the denominator, there are two possibilities;

(i) Denominator which will factorize.

(ii) Denominator which will not factorize.

But if the degree of the numerator is not less than the degree of the denominator, then the fraction must be divided out before integration is attempted.

Rules for Partial Fractions

1) To each linear factor such as $(x-a)$ in the denominator, there corresponds one partial fraction in the form of $\frac{A}{(x-a)}$.

2) To each repeated linear factor such as $(x-a)^2$ in the denominator, there correspond two partial fractions in the form of $\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$.

4) To each quadratic factor such as $\left(ax^2+bx+c\right)$ (no real solutions) in the denominator, there corresponds one partial fraction in the form of $\dfrac{Mx+N}{ax^2+bx+c}$.

The unknown constants A, B, C, M, N etc., are found by:

(i) Substituting suitable values of x.
(ii) Comparing coefficients.

Important Integrals (revision)

1) $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

2) $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

3) $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

4) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$

Worked examples

1) Find

$$I = \int \frac{x-3}{x^2-x-2}\,dx .$$

Solution:

$$\frac{x-3}{x^2-x-2} = \frac{x-3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$= \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

Hence $\quad x-3 = A(x+1) + B(x-2).$

Put $x = 2:$ $\quad (-1) = A.3 \Rightarrow A = -\dfrac{1}{3}.$

Put $x = -1:$ $\quad (-4) = B.(-3) \Rightarrow B = \dfrac{4}{3}.$

$$I = \int \frac{\left(-\frac{1}{3}\right)}{x-2}\,dx + \int \frac{\left(\frac{4}{3}\right)}{x+1}\,dx$$

$$= -\frac{1}{3}\ln(x-2) + \frac{4}{3}\ln(x+1) + C.$$

2) Find $J = \displaystyle\int \frac{1}{x(x+2)^2}\,dx.$

Solution:

The denominator has one linear factor x and one repeated linear factor $(x+2)^2$.

Let $\quad \dfrac{1}{x(x+2)^2} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$

$$= \frac{A(x+2)^2 + Bx(x+2) + Cx}{x(x+2)^2}.$$

Hence $\quad 1 = A(x+2)^2 + Bx(x+2) + Cx$

$$= (A+B)x^2 + (4A+2B+C)x + 4A.$$

Put $x = -2:$ $\quad 1 = C(-2) \Rightarrow C = -\dfrac{1}{2}.$

Put $x = 0:$ $\quad 1 = A(4) \Rightarrow A = \dfrac{1}{4}.$

Compare $x^2:$ $\quad 0 = A + B \Rightarrow B = -A = -\dfrac{1}{4}.$

So $$\frac{1}{x(x+2)^2} = \frac{\left(\frac{1}{4}\right)}{x} - \frac{\left(\frac{1}{4}\right)}{x+2} - \frac{\left(\frac{1}{2}\right)}{(x+2)^2},$$

and

$$J = \frac{1}{4}\int \frac{dx}{x} - \frac{1}{4}\int \frac{dx}{x+2} - \frac{1}{2}\int \frac{dx}{(x+2)^2}$$

$$= \frac{1}{4}\ln x - \frac{1}{4}\ln(x+2) + \frac{1}{2(x+2)} + C$$

$$= \frac{1}{4}\{\ln x - \ln(x+2)\} + \frac{1}{2(x+2)} + C$$

$$= \frac{1}{4}\ln\left(\frac{x}{x+2}\right) + \frac{1}{2(x+2)} + C.$$

3) Find $K = \int \frac{x^3}{(x-2)}\,dx.$

Solution:

The degree of the numerator is greater than the degree of the denominator!

We must divide out first:

$$\frac{x^3}{x-2} = \left(x^2 + 2x + 4\right) + \frac{8}{x-2}.$$

That we obtain

$$K = \int\left(x^2 + 2x + 4 + \frac{8}{x-2}\right)dx$$

$$= \frac{x^3}{3} + x^2 + 4x + 8\ln(x-2) + C.$$

4) Find $R = \int \frac{10 - 11x}{\left(x^2+1\right)(x-4)}\,dx.$

Solution:

There are one linear and one quadratic factor.

Let $\dfrac{10-11x}{\left(x^2+1\right)(x-4)} = \dfrac{A}{x-4} + \dfrac{Bx+C}{x^2+1}$

$$= \frac{A\left(x^2+1\right)+(Bx+C)(x-4)}{(x-4)\left(x^2+1\right)}.$$

Hence: $10-11x = A\left(x^2+1\right)+(Bx+C)(x-4)$

$$= (A+B)x^2+(C-4B)x+(A-4C).$$

Put $x=4$: $(-34) = A.17 \Rightarrow A=-2.$

Put $x=0$: $10 = A-4C \Rightarrow 10=-2-4C \Rightarrow C=-3.$

Compare x^2: $0 = A+B \Rightarrow 0=-2+B \Rightarrow B=2.$

So $\dfrac{10-11x}{\left(x^2+1\right)(x-4)} = \dfrac{-2}{x-4} + \dfrac{2x-3}{x^2+1}$

and

$$R = \int\left(\frac{-2}{x-4}+\frac{2x-3}{x^2+1}\right)dx$$

$$= \int\left(\frac{-2}{x-4}+\frac{2x}{x^2+1}-\frac{3}{x^2+1}\right)dx$$

$$= -2\ln(x-4)+\ln\left(x^2+1\right)-3\arctan x + C$$

$$= \ln\left(\frac{x^2+1}{(x-4)^2}\right)-3\arctan x + C.$$

Exercises

1) $\int \frac{x+1}{\left(x^2+4x-5\right)}dx$

2) $\int \frac{2x}{(x-2)(x+3)}dx$

3) $\int \frac{x-4}{x(x+1)}dx$

4) $\int \frac{5x+2}{(x-2)^2(x+1)}dx$

5) $\int \frac{2x-5}{\left(x^2-5x+6\right)}dx$

6) $\int \frac{x+2}{\left(x^2+2x-8\right)}dx$

7) $\int \frac{x-3}{x\left(x^2+1\right)}dx$

8) $\int \frac{x^2-8x+5}{(2x+1)\left(x^2+9\right)}dx$

9) $\int \frac{x-1}{\left(3x^2-11x+10\right)}dx$

10) $\int \frac{x+6}{x\left(x-3\right)\left(x+1\right)}dx$

• Integration by parts

From the product rule for differentiation

$\frac{d}{dx}(uv) = \left(\frac{du}{dx}\right)v + u\left(\frac{dv}{dx}\right)$, integrating both sides with respect to x, we obtain

$uv = \int \left(\frac{du}{dx}\right)vdx + \int u\left(\frac{dv}{dx}\right)dx$,

or $\int uv'dx = uv - \int u'vdx$,

i.e. $\int udv = uv - \int vdu$ $\rightarrow$ this is the formula for "***Integration by parts***".

"***Integration by parts***" will usually "work" for:

$$\int x^n.\begin{Bmatrix}\ln x\\ \arctan x\\ \arcsin x\\ \arccos x\end{Bmatrix}dx\,; \quad \int x^n.\begin{Bmatrix}e^x\\ \cos x\\ \sin x\end{Bmatrix}dx\,; \quad \int \sqrt{a^2 \pm x^2}dx\,.$$

Worked examples

1) $I = \int x^2 \ln x dx = \frac{1}{3}\int \ln x dx^3$.

Let $\ln x = u$ and $x^3 = v$.

Thus

$$I = \frac{1}{3}\int \ln x dx^3$$
$$= \frac{1}{3}\left[x^3 \ln x - \int x^3 d \ln x\right]$$
$$= \frac{1}{3}\left[x^3 \ln x - \int x^3 (\ln x)' \, dx\right]$$
$$= \frac{1}{3}\left[x^3 \ln x - \int x^3 \frac{1}{x} dx\right]$$
$$= \frac{1}{3}\left[x^3 \ln x - \int x^2 dx\right]$$
$$= \frac{1}{3}\left[x^3 \ln x - \frac{x^3}{3}\right] + C.$$

2) $J = \int xe^{2x} dx$

$$= \frac{1}{2}\int xe^{2x} d\left(2x\right) = \frac{1}{2}\int xde^{2x}.$$

Let $x = u$ and $e^{2x} = v$.

Then

$$J = \frac{1}{2}\left[xe^{2x} - \int e^{2x} dx\right]$$
$$= \frac{1}{2}\left[xe^{2x} - \frac{1}{2}\int e^{2x} d2x\right]$$
$$= \frac{1}{2}\left[xe^{2x} - \frac{1}{2} e^{2x}\right] + C.$$

3) $K = \int \arctan x dx = x\arctan x - \int x d\arctan x$

$= x\arctan x - \int x(\arctan x)' dx$

$= x\arctan x - \int x\left(\frac{1}{x^2+1}\right)dx$

$= x\arctan x - \frac{1}{2}\int \frac{d\left(x^2+1\right)}{x^2+1}$

$= x\arctan x - \frac{1}{2}\ln\left(x^2+1\right) + C.$

4) $L = \int_0^1 \arcsin x dx = \left[x\arcsin x\right]_0^1 - \int_0^1 x d\arcsin x.$

$= \left(1.\arcsin 1 - 0\right) - \int_0^1 x\left(\arcsin x\right)' dx$

$= 1.\frac{\pi}{2} - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx$

$= \frac{\pi}{2} - \frac{1}{2}\int_0^1 \frac{d\left(x^2-1\right)}{\left(1-x^2\right)^{\frac{1}{2}}}$

$= \frac{\pi}{2} + \frac{1}{2}\int_0^1 \frac{d\left(-x^2+1\right)}{\left(1-x^2\right)^{\frac{1}{2}}}$

$= \frac{\pi}{2} + \frac{1}{2}\left.\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^1 = \frac{\pi}{2} + (0-1) = -\frac{\pi}{2}.$

Exercises

1) $\int x\sin 2x dx$ **2)** $\int x^3 \ln x dx$ **3)** $\int xe^{-x} dx$

4) $\int \frac{\arctan x}{x^2} dx$ **5)** $\int x^2 e^x dx$ **6)** $\int x \arctan x dx$

7) $\int x \arcsin x dx$ **8)** $\int x \cos 3x dx$ **9)** $\int e^{2x} \sin x dx$

10) $\int \sqrt{4 + x^2} dx$ **11)** $\int \frac{x}{\cos^2 x} dx$ **12)** $\int_0^1 x e^{3x} dx$

13) $\int_0^{\pi} (1 + x) \sin x dx$ **14)** $\int_1^{e^2} \ln x dx$ **15)** $\int_1^2 \ln(2x + 1) dx$

• Certain Trigonometric Integrals

I. Odd powers of sine or cosine

Method: Substitute the "other" function and use $\sin^2 \theta + \cos^2 \theta = 1$.

Example.

Find $I = \int \sin^5 \theta d\theta$.

Let $u = \cos \theta$, $du = -\sin \theta d\theta$.

Also $\sin^4 \theta = \left(\sin^2 \theta\right)^2 = \left(1 - \cos^2 \theta\right)^2 = \left(1 - u^2\right)^2$.

Then

$$I = \int \sin^4 \theta \sin \theta d\theta$$

$$= \int \left(1 - u^2\right)^2 \left(-du\right)$$

$$= -\int \left(1 - 2u^2 + u^4\right) du$$

$$= -\left(u - \frac{2}{3} u^3 + \frac{u^5}{5}\right) + C$$

$$= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{\cos^5\theta}{5} + C\,.$$

II. Even powers of sine or cosine

Method: Reduce to a sum of terms by means of repeated application of the double (or half) angle formulas:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1}{2}(1-\cos 2\theta)$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} = \frac{1}{2}(1+\cos 2\theta)$$

Example.

Find $J = \int \sin^4 2x dx$.

$$J = \int \left\{\frac{1}{2}(1-\cos 4x)\right\}^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 4x + \cos^2 4x\right)dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 4x + \frac{1}{2}(1+\cos 8x)\right)dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 4x + \frac{1}{2}\cos 8x\right)dx$$

$$= \frac{1}{4}\left(\frac{3x}{2} - \frac{2\sin 4x}{4} + \frac{1}{2}\frac{\sin 8x}{8}\right) + C$$

$$= \frac{1}{64}\left(24x - 8\sin 4x + \sin 8x\right) + C\,.$$

III. Integrands of the form
$\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha\cos\beta$

Method: Use the "factor formulas" to split the product into a sum instead.

Example.

Find $K = \int \sin 5x \cos 3x dx$.

Using the trigonometric identity

$$\sin 5x \cos 3x = \frac{1}{2}(\sin 8x + \sin 2x),$$

we have:

$$K = \frac{1}{2}\int (\sin 8x + \sin 2x)\, dx = \frac{1}{2}\left(\frac{-\cos 8x}{8} - \frac{\cos 2x}{2}\right) + C.$$

Exercises

1) $\int \sin^2(3x)\, dx$ **2)** $\int \cos^3 x dx$ **3)** $\int_0^{\pi} \sin^4 x dx$

4) $\int \sinh^2(2x)\, dx$ **5)** $\int \sin 2x \cos x dx$

6) $\int_0^{\frac{\pi}{4}} \cos 3\theta \cos \theta d\theta$ **7)** $\int_0^{\pi} \cos 2x \cos 3x dx$

8) $\int \sin 3\theta \cos^2 \theta d\theta$ **9)** $\int \sin \theta \sin 3\theta d\theta$

• Improper Integral

We defined $\int_a^b f(x)\, dx$ with the following restrictive conditions:

a) a and b are finite numbers;

b) $f(x)$ is continuous on $[a,b]$.

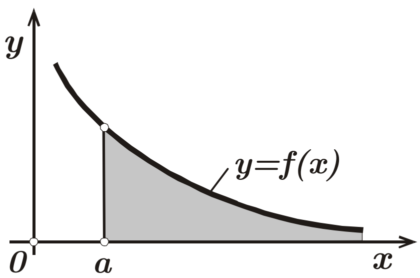
But in the applied problems it is necessary to weaken these restrictive conditions.

The following two types of integrals are called **improper** , while the integrals with a) and b) conditions – **proper.**

I) Improper Integrals with infinite limits.

Let $f(x)$ be continuous on $[a,\infty)$ with $f(x)\geq 0$ for all $x\geq a$.

Definition1. $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)dx$$



If this limit exists, it is said that the improper integral ***converges .***

If this limit doesn't exist, it is said that the improper integral ***diverges .***

In the same way we can define $\int_{-\infty}^{b} f(x)\,dx$.

By using the Fundamental Theorem (Leibniz-Newton's formula)

we shall make use of the notation

$$\int_a^\infty f(x)\,dx = F(x)\Big|_a^\infty = F(\infty)-F(a), \ \left(F'(x)=f(x)\right).$$

Example1.

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}(\ln x)\Big|_1^t = \lim_{t\to\infty}(\ln t-\ln 1)=\infty$$

(diverge)

Example2.

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2} dx = \lim_{t\to\infty} \left(-\frac{1}{x}\Big|\right)_1^t = \lim_{t\to\infty} \left(-\frac{1}{t} - (-1)\right) = 1$$

(conv.)

Example3.

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \{\arctan x\}|_t^0$$

$$= \arctan 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \text{(conv.)}$$

Conclusion: $\int_a^{\infty} \frac{1}{x^k}\,dx$ converges $\Leftrightarrow$ $k > 1$.

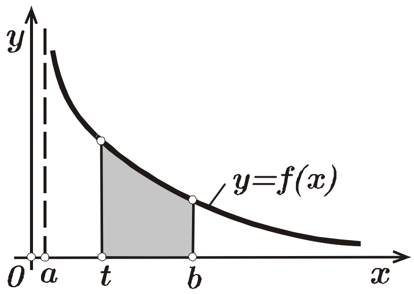
Definition2. The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined by

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx, \quad c \in \left(-\infty, +\infty\right)$$

II) Improper Integrals with finites limits.

Let $f(x)$ be continuous on $(a, b]$, but $\lim_{x\to a^+} f(x) = \pm\infty$.

Definition 3. $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x) dx$.

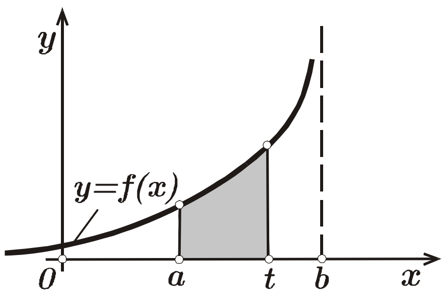


Exp. $\int_0^4 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^4 = 2.2 = 4.$

If $f(x)$ is continuous on $[a,b)$,

but $\lim_{x\to b^-} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.$$



Exp. $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{x}{\sqrt{1-x^2}}\,dx$

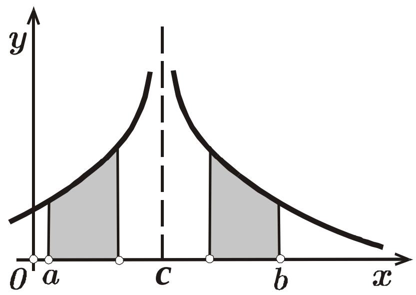
$$= \lim_{t\to 1^-}\frac{1}{2}\int_0^t \frac{dx^2}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\left\{-\sqrt{1-x^2}\Big|_0^t\right\}$$

$$= \lim_{t\to 1^-}\left\{-\sqrt{1-t^2}+1\right\} = 1.$$

If $f(x)$ is continuous on $[a,c)\cup(c,b]$,

and $\lim_{x\to c} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \qquad c \in \left(a,b\right).$$



Example.

$$\int_0^3 (x-2)^{-\frac{4}{3}}\,dx =$$

$$= \lim_{t\to 2^-} \int_0^t (x-2)^{-\frac{4}{3}}dx + \lim_{t\to 2^+} \int_t^3 (x-2)^{-\frac{4}{3}}dx$$

$$= \lim_{t\to 2^-} \left\{ -3(x-2)^{-\frac{1}{3}} \Big|_0^t \right\} + \lim_{t\to 2^+} \left\{ -3(x-2)^{-\frac{1}{3}} \Big|_t^3 \right\}$$

$$= \lim_{t\to 2^-} \left\{ \frac{-3}{\sqrt[3]{t-2}} + \frac{3}{\sqrt[3]{-2}} \right\} + \lim_{t\to 2^+} \left\{ -3 + \frac{3}{\sqrt[3]{t-2}} \right\}$$

$$= \infty + \infty = \infty.$$

Sometimes, we need to determine only whether an improper integral converges or diverges. In such cases we can use the Comparison test.

Comparison test

Let $f(x)$ and $g(x)$ continuous on $[a,+\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$.

Then

(I) If $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$.

(II) If $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$.

Example1.

$$I = \int_1^\infty \frac{1}{x^2 + \cos^2 x}\,dx \text{ converges,}$$

because we have $\dfrac{1}{x^2 + \cos^2 x} \le \dfrac{1}{x^2}, \quad x \ge 1$

and because $\displaystyle\int_1^\infty \frac{dx}{x^2} = 1$ converges.

Example2.

$J = \displaystyle\int_1^e \frac{dx}{x \ln^3 x}$ diverges, because $\displaystyle\int_1^e \frac{dx}{x \ln x} = \infty$ (div.)

Example3.

$$K = \int_1^\infty e^{-\left(x^2+1\right)}dx.$$

From $\left(1+x^2\right) > x$ for $x \geq 1$ and $e^{-\left(x^2+1\right)} < e^{-x}$, where e^{-x} is a decreasing function,

we have

$$\int_1^{\infty} e^{-x} dx = \lim_{t\to\infty} \int_1^{t} e^{-x} dx = \lim_{t\to\infty} \left(-e^{-x}\right)\Big|_1^t$$

$$= \lim_{x\to\infty} \left(\frac{1}{e} - e^{-t}\right) = \frac{1}{e} - 0 = \frac{1}{e}.$$

So $K = \int_1^{\infty} e^{-\left(x^2+1\right)} dx$ converges.

- **Exercises**

1) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \arctan x \Big|_{-\infty}^{+\infty}$

$$= \lim_{x\to+\infty} \{\arctan x\} - \lim_{x\to-\infty} \{\arctan x\}$$

$$= \left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) = \pi.$$

2) $\int_2^{\infty} \frac{1}{x} dx$ **3)** $\int_2^{\infty} \frac{1}{(x+3)^2} dx$ **4)** $\int_1^{\infty} \frac{1}{x^2+2} dx$

5) $\int_0^{\infty} e^{-x} dx$ **6)** $\int_{-\infty}^{2} e^{3x} dx$ **7)** $\int_1^{\infty} \frac{x}{\sqrt{1+x^2}} dx$

8) $\int_0^{2} (x-1)^{\frac{-4}{3}} dx$ **9)** $\int_2^{\infty} \frac{1}{x^2+x-2} dx$ **10)** $\int_{-\infty}^{\infty} \frac{2x}{x^2+1} dx$

11) $\int_2^{\infty} \frac{1}{x \ln x} dx$ **12)** $\int_2^{\infty} \frac{1}{x^2+1} dx$ **13)** $\int_0^{\infty} x e^{-x} dx$

14) $\int\limits_{-\infty}^{\infty} xe^{-x^2}\,dx$ **15)** $\int\limits_{-1}^{1} \frac{1}{x}\,dx$ **16)** $\int\limits_{0}^{3} \frac{1}{(x-1)}\,dx$

17) $\int\limits_{0}^{\frac{\pi}{2}} \tan x dx$ **18)** $\int\limits_{1}^{e} \frac{1}{x\sqrt[3]{\ln x}}\,dx$ **19)** $\int\limits_{0}^{4} \frac{1}{\sqrt{16-x^2}}\,dx$

20) $\int\limits_{0}^{1} x\ln x dx$ **21)** $\int\limits_{e^2}^{\infty} \frac{1}{x\ln^3 x}\,dx$ **22)** $\int\limits_{1}^{\infty} \frac{\arctan x}{x^2}\,dx$

23) $\int\limits_{1}^{2} \frac{3x}{\left(2x^2-1\right)^3}\,dx$ **24)** $\int\limits_{0}^{1} \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx$ **25)** $\int\limits_{0}^{1} \sqrt{1+x^2}dx$

26) $\int\limits_{0}^{3} \frac{1}{(x-1)^2}\,dx$ **27)** $\int\limits_{0}^{\frac{\pi}{4}} \frac{1+3\sin^2 x}{\cos^2 x}\,dx$ **28)** $\int\limits_{0}^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos x}}\,dx$

29) $\int\limits_{0}^{\frac{\pi}{2}} x\cos x\,dx$ **30)** $\int\limits_{0}^{1} \frac{x}{\left(5+x^2\right)^2}\,dx$ **31)** $\int\limits_{0}^{\frac{\pi}{4}} \frac{\sin x}{3+\sin^2 x}\,dx$

References

1. *Berkey,D.D., Calculus, Boston University, SAUNDERS College Publishing.*

2. *L.Bostock; S.Chandler, Sc.Matemathics-The Core Course for A-Level.*